J. Fluid Mech. (1964), vol. 20, part 3, pp. 481–499
Printed in Great Britain

The radial spread of a liquid jet over a horizontal plane

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(Received 13 August 1963 and in revised form 6 May 1964)

When a smooth jet of water falls vertically on to a horizontal plane, it spreads out radially in a thin layer bounded by a circular hydraulic jump, outside which the depth is much greater. The motion in the layer is studied here by means of boundary-layer theory, both for laminar and for turbulent flow, and relations are obtained for the radius of the hydraulic jump. These relations are compared with experimental results. The analogous problems of two-dimensional flow are also treated.

1. Introduction

It is a familiar observation that when a smooth jet of water falls vertically from a tap on to a horizontal plane, such as the bottom of an empty sink, the water spreads out in a thin layer until a sudden increase of depth occurs. This is an hydraulic jump, or standing wave, the stationary counterpart of a tidal bore. The formation of the thin layer and the circular jump was noticed by Rayleigh (1914), who derived the properties of bores and jumps. Rayleigh's analysis refers to flow along a channel of constant breadth, and assumes the speed ahead of the wave to be uniform. In the present case the flow in the thin layer is radial and strongly influenced by viscosity, but the principles of momentum and continuity apply at the jump as in Rayleigh's theory.

Since the central layer of fluid is thin, it is natural to apply the ideas of boundary-layer theory in order to discuss the motion. A necessary condition for this approach to be valid is that the Reynolds number of the impinging jet shall be large. The depth is observed to be much greater on the outside of the jump than on the inside, and hence the condition at the jump may be simplified. This observation is equivalent to the statement that the Froude number for the flow outside the jump is small. It will further be assumed that the radius of the standing wave is much greater than that of the incident jet. No account is taken of the structure of the hydraulic jump, or surface tension effects.

The first problem, treated in § 2, is the simple case in which viscosity is completely ignored. It is more realistic, however, to assume that a boundary layer will grow on the plane from the central stagnation point, and that this boundary layer will gradually absorb the whole of the flow until the whole layer is a boundary layer. Thus, for large values of the radial distance r, a similarity solution of the laminar boundary-layer equations may be sought, and this is obtained in § 3. This similarity solution is found to involve the combination (r^3+l^3), where l is an arbitrary constant length. In § 4 the value of l^3 appropriate to the flow considered is estimated from an approximate solution of the Pohlhausen type for the growth of the boundary layer.

In § 5 the principle of momentum is applied at the hydraulic jump (neglecting its radial width), and a relation is derived for r_1, the radius of the jump. This relation involves the depth d of the water outside the wave, and d is regarded as prescribed by the conditions of outflow at a great distance. The other physical quantities appearing are Q (volume rate of flow), a (radius of the jet), ν (kinematic viscosity) and g (gravitational acceleration). From these 6 quantities 4 dimensionless parameters can be formed. It is assumed that $R = Q/\nu a$ (jet Reynolds number) is large and that $Q^2/r_1^2 g d^3$ (proportional to the Froude number outside the wave) appears only in a small correction. When this correction is ignored the relation connects $r_1 d^2 g a^2/Q^2$ with $(r_1/a)R^{-\frac{1}{3}}$. In addition, the radial width of the jump must be small compared with r_1, and since the width may be of the order of $5d$ this requires that d/r_1 shall be small.

In this application of boundary-layer theory the gravitational pressure gradient, due to the variation in height of the free surface, is neglected. An earlier theory, due to Kurihara (1946) and Tani (1948), regards the hydraulic jump as a separation of the flow, induced by the gravitational pressure gradient. Consequently some investigations are made in § 6 of the conditions in which this neglect of gravity may be justified. These conditions are adequately fulfilled in the experiments described in § 8.

The treatment so far described applies only to laminar flow. To deal with turbulent flow use has been made of the hypothesis, introduced by Glauert (1956), of an eddy viscosity which varies across the boundary layer like u^6, where u is the radial velocity. A solution analogous to that of §§ 3–5 is given in § 7.

Experiments were made in an attempt to verify the theoretical predictions, and are described in § 8. Although the results show a wide scatter, they appear to be consistent with the assumptions of the theory.

Finally, a brief treatment is given in § 9 of the analogous problems of two-dimensional flow.

2. Inviscid theory

When viscosity is ignored, the motion produced by a round jet falling vertically on to a horizontal plane is one of potential flow with free streamlines. Methods for the solution of problems of this type are described by Birkhoff & Zarantonello (1957). When r, the distance from the axis of the jet, is large compared with a, the radius of the impinging jet, the depth h of the fluid on the plane is small and the motion is almost radial with speed U_0, the speed with which the jet strikes the plane. Hence the volume rate of flow is

$$Q = \pi a^2 U_0 = 2\pi r h U_0, \tag{1}$$

so that

$$h = a^2/2r. \tag{2}$$

The condition to be applied at the jump (due originally to Bélanger 1838) is that the thrust of the pressure is equal to the rate at which momentum is destroyed. The depth on the inside of the jump is given by (2) with $r = r_1$, the radius of the standing wave. If d is the depth outside, the thrust of the pressure per unit length of wave is $\frac{1}{2}\rho g(d^2-h^2)$, where ρ is the density. The speed of flow is U_0 inside the jump, and outside it is

$$U_1 = Q/2\pi r_1 d. \tag{3}$$

The rate of destruction of momentum per unit length of wave is therefore $\rho(U_0^2 h - U_1^2 d)$. Thus

$$\tfrac{1}{2}g(d^2-h^2) = \left(\frac{Q}{2\pi r_1}\right)^2\left(\frac{1}{h}-\frac{1}{d}\right). \tag{4}$$

When $h \ll d$, this reduces to

$$\tfrac{1}{2}gd^2 = Q^2/4\pi^2 r_1^2 h = Q^2/2\pi^2 r_1 a^2,$$

that is

$$r_1 d^2 g a^2/Q^2 = 1/\pi^2. \tag{5}$$

A better approximation is to neglect only $(h/d)^2$ in (4), so that the pressure thrust inside the wave is ignored but the momentum outside is included. This gives

$$\frac{r_1 d^2 g a^2}{Q^2} + \frac{a^2}{2\pi^2 r_1 d} = \frac{1}{\pi^2}. \tag{6}$$

The ratio of the second term of (6) to the first is $2U_1^2/gd$ so that, if the correction term is to be small, the Froude number of the outer flow must be small.

Equation (4) can be solved exactly, to give

$$\frac{r_1 d^2 g a^2}{Q^2} = \frac{1}{\pi^2} - \frac{gda^4}{2Q^2}. \tag{7}$$

However, in the solutions considered later, the equations corresponding to (4) cannot be treated so simply, and therefore results analogous to (6) will be derived by neglecting $(h/d)^2$.

3. Similarity solution of the boundary-layer equations

According to the boundary-layer approximations the flow in the thin layer satisfies the equations

$$\partial(ru)/\partial r + \partial(rw)/\partial z = 0, \tag{8}$$

$$u(\partial u/\partial r) + w(\partial u/\partial z) = \nu(\partial^2 u/\partial z^2), \tag{9}$$

with the conditions

$$u = w = 0 \quad \text{at} \quad z = 0, \tag{10}$$

$$\partial u/\partial z = 0 \quad \text{at} \quad z = h(r), \tag{11}$$

$$2\pi r \int_0^{h(r)} u\,dz = Q. \tag{12}$$

Here r, z are cylindrical co-ordinates, with z measured vertically upwards from the plate, and u, w are the corresponding velocity components. In equation (9) the gravitational pressure gradient $(-\rho g\,dh/dr)$ has been ignored. Equation (11) asserts that the shearing stress falls to zero at the free surface $z = h(r)$, since the viscosity of air is negligible, and (12) is the condition of constant volume flux.

In this section a similarity solution will be derived by assuming that

$$u = U(r) f(\eta), \tag{13}$$

$$\eta = z/h(r), \tag{14}$$

where $U(r)$ is the speed at the free surface. Then from (10) and (11)

$$f(0) = 0, \quad f(1) = 1, \quad f'(1) = 0, \tag{15}$$

and from (12)

$$Q = 2\pi r U h \int_0^1 f(\eta)\, d\eta. \tag{16}$$

Hence rUh is constant, and (8) then leads to

$$w = Uh'\eta f(\eta). \tag{17}$$

The equation of motion (9) now reduces to

$$\nu f''(\eta) = h^2 U' f^2(\eta),$$

from which it follows that h^2U' is constant. Also $f''(\eta) \leqslant 0$, since the shearing stress is greatest at the plate, and it is convenient to write

$$h^2 U' = -\tfrac{3}{2} c^2 \nu, \tag{18}$$

where c is a number. Then $2f'' = -3c^2 f^2$, which, from (15), may be integrated to $f'^2 = c^2(1-f^3)$. Since $f' \geqslant 0$,

$$c\eta = \int_0^f (1-x^3)^{-\frac{1}{2}}\, dx. \tag{19}$$

The condition $f(1) = 1$ now gives

$$c = \int_0^1 (1-x^3)^{-\frac{1}{2}}\, dx = \frac{1}{3}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})} = 1{\cdot}402. \tag{20}$$

Then

$$\int_0^1 f(\eta)\, d\eta = c^{-1}\int_0^1 f(1-f^3)^{-\frac{1}{2}}\, df = \frac{1}{3c}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{6})} = \frac{2\pi}{3\sqrt{3}c^2}. \tag{21}$$

Consequently (16) gives

$$rUh = 3\sqrt{3}c^2 Q/4\pi^2. \tag{22}$$

The only conditions on $U(r)$, $h(r)$ necessary for the similarity solution are (18) and (22). The general solution of these equations is

$$U(r) = \frac{27c^2}{8\pi^4}\,\frac{Q^2}{\nu(r^3+l^3)}, \tag{23}$$

$$h(r) = \frac{2\pi^2}{3\sqrt{3}}\,\frac{\nu(r^3+l^3)}{Qr}, \tag{24}$$

where l is an arbitrary constant length.

In the actual flow this similarity solution can only be expected to hold when r is sufficiently large for the conditions in the incident jet to have lost their influence. The value of l, however, depends on these conditions, and must be found by consideration of the growth of the boundary layer from the point of impact of the jet. A method for the estimation of l will be described in §4.

The velocity profile in the similarity solution is given by (19), which can also be expressed by means of Jacobian elliptic functions (Neville 1944) as

$$f(\eta) = \sqrt{3}+1-\frac{2\sqrt{3}}{1+\mathrm{cn}\,\{3^{\frac{1}{4}}c(1-\eta)\}}, \tag{25}$$

where the modulus is $\sin 75°$. Hence, in terms of the elliptic integral $F(\theta)$ with this modulus,

$$\left.\begin{aligned} \nu(r^3+l^3)\,u/Q^2 &= (27c^2/8\pi^4)\,(1-\surd 3\tan^2\tfrac{1}{2}\theta),\\ Qrz/\nu(r^3+l^3) &= (2\pi^2/3\surd 3)\,\{1-3^{-\frac{1}{4}}c^{-1}F(\theta)\}.\end{aligned}\right\} \qquad (26)$$

$\theta = 0$ corresponds to the free surface and $\theta = \cos^{-1}(2-\surd 3) \simeq 74\tfrac{1}{2}°$ to the surface of the plate. The profile (25) is shown in figure 1.

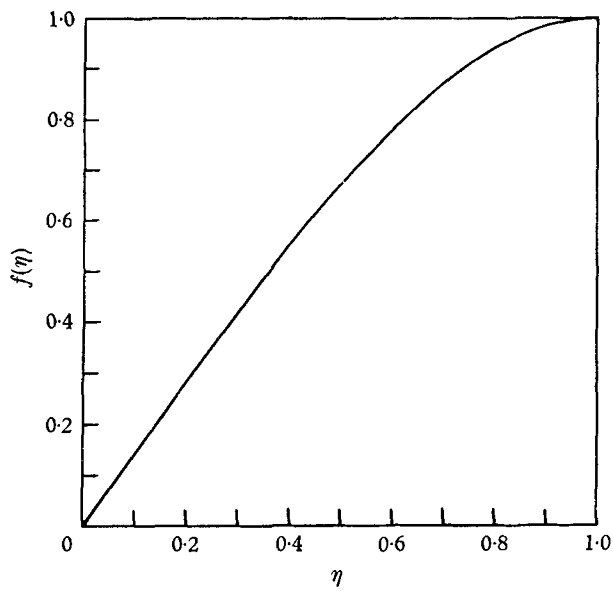

FIGURE 1. The velocity distribution function.

It is of interest to observe that the analogous problems of the wall jet (Glauert 1956, 1958) and the radial free jet (Squire 1955) also yield similarity solutions in which the combination (r^3+l^3) occurs as in equations (23) and (24). In this connexion see Riley (1961, 1962).

4. General approximate solution

As already remarked in § 1, the boundary layer grows from the stagnation point on the axis of the jet until it absorbs the whole of the flow. In fact, four regions of flow may be distinguished, though they pass continuously into one another.

(i) When $r = O(a)$, the speed outside the boundary layer rises rapidly from 0 at the stagnation point to U_0, and the boundary-layer thickness is $O(\nu a/U_0)^{\frac{1}{2}}$ (Homann 1936).

(ii) For greater values of r the speed outside the boundary layer remains almost constant, equal to U_0, as the fluid here is unaffected by the viscous stresses. The boundary-layer flow in this region is therefore given by equations (8) and (9) with the conditions

$$u = w = 0 \quad \text{at} \quad z = 0, \quad u \to U_0 \quad \text{as} \quad z \to \infty. \qquad (27)$$

If $r \gg a$, so that the conditions in region (i) do not affect the flow, a solution of these equations can be found in which

$$u = U_0 f_1'(\eta_1), \quad \eta_1 = (3U_0/2\nu r)^{\frac{1}{2}} z, \tag{28}$$

where

$$\left.\begin{aligned} & f_1''' + f_1 f_1'' = 0, \\ & f_1(0) = f_1'(0) = 0, \quad f_1'(\infty) = 1. \end{aligned}\right\} \tag{29}$$

Thus the velocity distribution has the Blasius flat-plate profile, and the boundary-layer thickness is $O(\nu r/U_0)^{\frac{1}{2}}$. It also follows that

$$\left.\begin{aligned} \left(\frac{\nu r}{U_0}\right)^{\frac{1}{2}} \frac{1}{U_0} \left(\frac{\partial u}{\partial z}\right)_{z=0} &= f_1''(0)\sqrt{\tfrac{3}{2}} = 0{\cdot}575, \\ H = \delta_1/\delta_2 &= 2{\cdot}59, \end{aligned}\right\} \tag{30}$$

where δ_1, δ_2 are the displacement and momentum thicknesses.

(iii) When the viscous stresses become appreciable right up to the free surface the whole flow is of boundary-layer type. The velocity profile changes as r increases, from the Blasius type (28) to the similarity profile (25).

(iv) Ultimately the way in which the flow originated becomes unimportant, and the similarity solution of § 3 is valid, with an appropriate choice of the length l. However, as noted earlier, the value of l depends on the development of the flow in the inner regions. The order of magnitude of l can be found by elementary considerations, as follows.

In region (ii) the total depth h of the flow is $O(a^2/r)$, as in equation (2). Since the boundary-layer thickness δ is $O(\nu r/U_0)^{\frac{1}{2}}$, δ becomes comparable with h when $r = O(aR^{\frac{1}{3}})$, where $R = Q/\nu a$ is the Reynolds number of the incident jet. The solution (28) is valid only when $r \gg a$, so that R must be large.

In region (iii), $r = O(aR^{\frac{1}{3}})$ and the speed U at the free surface is $O(U_0)$. Also, since there is a transition to the similarity solution, it follows from (23) that $r^3 + l^3 = O(Q^2/\nu U_0) = O(a^3 R)$. Thus $l = O(aR^{\frac{1}{3}})$, of the same order as the length scale for the radial development of the boundary layer in regions (ii) to (iv), and since R is large, $l \gg a$.

It has been shown that the velocity profile must actually change through the transition region (iii) from the Blasius profile to the similarity profile. These profiles are not very different, however. A satisfactory approximation may therefore be expected if the Kármán–Pohlhausen method is applied with a constant velocity profile

$$u = U(r) f(z/\delta), \tag{31}$$

where $f(\eta)$ is the similarity-profile function, defined by (19) or (25), and δ is the boundary-layer thickness. This has the effect of suppressing the region (iii) in which the velocity profile changes. Hence on this approximation there is unretarded fluid present when $r < r_0$ (say) so that $\delta < h$ and $U(r) = U_0$, whereas for $r > r_0$ there is a similarity solution as in § 3 with $\delta = h$ and $U(r) < U_0$. In the region $r < r_0$ an approximation to the Blasius type of solution (28) will be derived, and r_0 is given by the condition $\delta = h$, so that the whole flow passes through the boundary layer.

The momentum integral equation for the flow in the region $r < r_0$ is

$$\left(\frac{d}{dr}+\frac{1}{r}\right)\int_0^\delta (U_0 u - u^2)\,dz = \nu\left(\frac{\partial u}{\partial z}\right)_{z=0}, \tag{32}$$

since in the unretarded fluid $u = U_0$. When u is given by the approximate profile (31) with $U(r) = U_0$, equation (32) becomes

$$\frac{2(\pi - c\sqrt{3})}{3\sqrt{3}c^2}\,U_0^2\left(\frac{d\delta}{dr}+\frac{\delta}{r}\right) = \frac{\nu U_0 c}{\delta}.$$

Hence
$$r^2\delta^2 = \frac{c^3\sqrt{3}}{\pi - c\sqrt{3}}\,\frac{\nu r^3}{U_0} + C, \tag{33}$$

where C is a constant. If (33) is to remain valid as $r \to 0$, then $C = 0$. If allowance were made for the fact that the main stream actually has a stagnation point at $r = 0$ then, as already shown, $\delta^2 = O(\nu a/U_0)$ when $r = O(a)$, so that $C = O(\nu a^3/U_0)$. Thus C would be $O(a^3/r^3)$ relative to the other terms of (33), and could therefore be neglected when $r \gg a$. Consequently when $a \ll r < r_0$,

$$\delta^2 = \frac{\pi\sqrt{3}c^3}{\pi - c\sqrt{3}}\,\frac{\nu r a^2}{Q}. \tag{34}$$

It follows that on the present approximation

$$\left.\begin{aligned} \left(\frac{\nu r}{U_0}\right)^{\frac{1}{2}}\frac{1}{U_0}\left(\frac{\partial u}{\partial z}\right)_{z=0} &= \left(\frac{\pi - c\sqrt{3}}{c\sqrt{3}}\right)^{\frac{1}{2}} = 0{\cdot}542,\\ H = \frac{\delta_1}{\delta_2} &= \frac{3\sqrt{3}c^2 - 2\pi}{2\pi - 2\sqrt{3}c} = 2{\cdot}76. \end{aligned}\right\} \tag{35}$$

Comparison of the approximate values (35) with the accurate values (30) shows that the errors are only about 6 %, which is adequate for the present purpose.

The boundary layer just absorbs the whole flow when $r = r_0$. Hence r_0 is given by the condition that the volume flux through the boundary layer reaches the value Q. As found in (22),

$$r_0 U_0 \delta(r_0) = (3\sqrt{3}c^2/4\pi^2)\,Q.$$

Since $\delta(r)$ is given by (34), this leads to

$$r_0^3 = \{9\sqrt{3}c(\pi - c\sqrt{3})/16\pi^3\}\,(Qa^2/\nu), \tag{36}$$

or
$$r_0 = 0{\cdot}3155 a R^{\frac{1}{3}}, \tag{37}$$

where R is the jet Reynolds number $Q/\nu a$, assumed large.

When $r < r_0$ the total depth h of the layer is given by the volume flux condition

$$2\pi r\left\{U_0\delta\int_0^1 f(\eta)\,d\eta + U_0(h-\delta)\right\} = Q.$$

Hence
$$h = (a^2/2r) + \{1 - (2\pi/3\sqrt{3}c^2)\}\,\delta, \tag{38}$$

where δ is given by (34).

The value of the length l in (23) and (24) can now be estimated on the present approximation. The free surface velocity $U(r)$ of (23) must be equal to U_0 when $r = r_0$. Consequently

$$l^3 = \{9\sqrt{3}c(3\sqrt{3}c - \pi)/16\pi^3\}\,(Qa^2/\nu),$$

that is
$$l = 0{\cdot}567 a R^{\frac{1}{3}}. \tag{39}$$

Thus on the present theory the depth h of the fluid is given by (38) for $r < r_0$ and by (24) for $r \geqslant r_0$, while the speed at the free surface is U_0 for $r < r_0$ and is given by (23) for $r \geqslant r_0$. Figure 2 shows the variation of U/U_0 and $(h/a)R^{\frac{1}{3}}$, with $(\delta/a)R^{\frac{1}{3}}$, as functions of $(r/a)R^{-\frac{1}{3}}$. It will be noticed that h has a minimum when $r = 2^{-\frac{1}{3}}l = 1{\cdot}43r_0$, and here $-U'(r)$ reaches its greatest value.

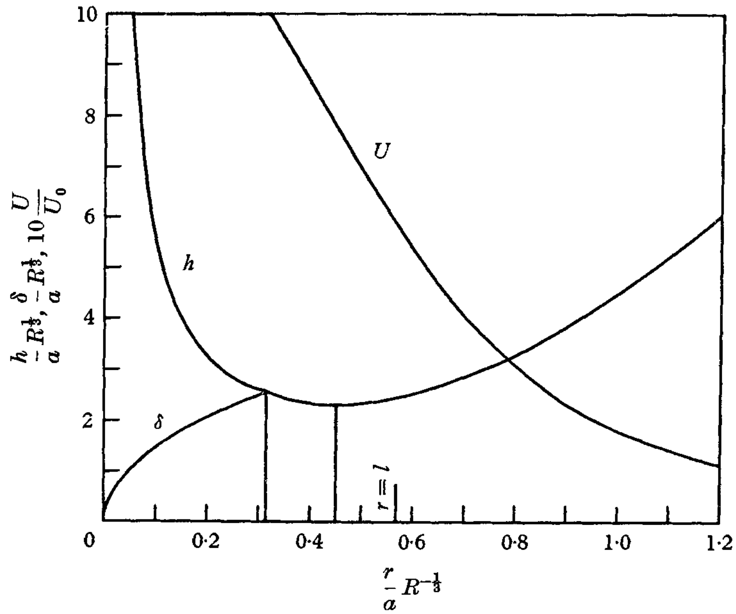

FIGURE 2. Variation of layer thickness and surface speed with radial distance (laminar flow).

5. Jump condition

The position, $r = r_1$, of the hydraulic jump is determined, as in § 2, by equating the rate of loss of momentum to the thrust of the pressure. This is legitimate provided that the width (measured radially) of the jump is small, so that skin friction can be ignored. As stated in § 1, this will hold if the depth d outside the jump is small compared with r_1. Owing to the complicated form of the expressions for h, it is undesirable to include the term $\frac{1}{2}\rho g h^2$, which is the pressure thrust on the inward side of the wave. This term, however, is only $O(h^2/d^2)$ compared with the thrust on the outward side, where the depth is d. Also the momentum outside the wave, which is $O(h/d)$ compared with that inside, will be included only approximately since it is assumed that the speed of flow U_1 immediately outside the jump is uniform, and therefore given by (3). The condition of momentum is thus

$$\tfrac{1}{2}\rho g d^2 = \rho \int_0^h u^2\,dz - \rho U_1^2 d,$$

so that

$$\frac{r_1 d^2 g a^2}{Q^2} + \frac{a^2}{2\pi^2 r_1 d} = \frac{2r_1 a^2}{Q^2}\int_0^h u^2\,dz. \tag{40}$$

It is necessary to evaluate the right-hand side of (40) separately for $r_1 \geqslant r_0$ and for $r_1 < r_0$, since the jump may occur at any point in the development of the boundary layer. When $r_1 \geqslant r_0$

$$\int_0^h u^2 dz = U^2 h \int_0^1 f^2(\eta)\, d\eta = \frac{27\sqrt{3}c^3}{16\pi^6} \frac{Q^3}{\nu r_1(r_1^3 + l^3)},$$

and (40) becomes

$$\left.\begin{aligned} \frac{r_1 d^2 g a^2}{Q^2} + \frac{a^2}{2\pi^2 r_1 d} &= \frac{27\sqrt{3}c^3}{8\pi^6}\left[\frac{r_1^3}{a^3} R^{-1} + \frac{9\sqrt{3}c(3\sqrt{3}c - \pi)}{16\pi^3}\right]^{-1}, \\ &= 0{\cdot}01676\{(r_1/a)^3 R^{-1} + 0{\cdot}1826\}^{-1}, \end{aligned}\right\} \quad (41)$$

provided that $(r_1/a)R^{-\frac{1}{3}} \geqslant 0{\cdot}3155$. When $r_1 < r_0$

$$\begin{aligned} \int_0^h u^2 dz &= U_0^2 \delta \int_0^1 f^2(\eta)\, d\eta + U_0^2(h - \delta), \\ &= U_0^2\left(\frac{a^2}{2r_1} - \frac{2(\pi - c\sqrt{3})}{3\sqrt{3}c^2}\,\delta\right). \end{aligned}$$

In this case (40) takes the form

$$\left.\begin{aligned} \frac{r_1 d^2 g a^2}{Q^2} + \frac{a^2}{2\pi^2 r_1 d} &= \frac{1}{\pi^2} - \frac{4}{3\pi^2}\left(\frac{\pi^2 - c\sqrt{3}\pi}{c\sqrt{3}}\right)^{\frac{1}{2}}\left(\frac{r_1}{a} R^{-\frac{1}{3}}\right)^{\frac{3}{2}}, \\ &= 0{\cdot}10132 - 0{\cdot}1297\,(r_1/a)^{\frac{3}{2}} R^{-\frac{1}{2}}, \end{aligned}\right\} \quad (42)$$

where

$$(r_1/a)R^{-\frac{1}{3}} < 0{\cdot}3155.$$

The actual form of the results (41) and (42) depends on the approximation made in § 4 that the velocity profile could be taken as constant. Nevertheless, provided $(h/d)^2$ and $(a/r_1)^3$ are negligible, the quantity

$$\frac{r_1 d^2 g a^2}{Q^2} + \frac{a^2}{2\pi^2 r_1 d}$$

should be a function of $(r_1/a)R^{-\frac{1}{3}}$ only, and the leading terms of (41) and (42) should be correct for large and small values, respectively, of $(r_1/a)R^{-\frac{1}{3}}$. This dimensionless jump radius may take a wide range of values while the quantities $(h/d)^2$ and $(a/r_1)^3$ remain small.

6. Effect of the gravitational pressure gradient

Tani (1948), following Kurihara (1946), regarded the hydraulic jump as a separation of the flow from the plane, due to the adverse gravitational pressure gradient, so that equation (9) is replaced by

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -g\frac{dh}{dr} + \nu\frac{\partial^2 u}{\partial z^2}, \quad (43)$$

and r_1 is given by $(\partial u/\partial z)_{z=0} = 0$ at $r = r_1$. Tani assumed that $h = 0$ at $r = 0$.

This approach leads to a relation between r_1 and Q. Since the depth d outside the jump can be varied independently of Q (for instance, if the empty sink is allowed to fill gradually), and this produces a corresponding variation in r_1, the

Kurihara–Tani theory cannot provide a full explanation of the jump. Nevertheless, it is important to test whether the gravitational pressure gradient might be significant. Some information may be gathered from the Holstein–Bohlen (1940) parameter, which from (43) is

$$\lambda \equiv -\frac{\delta_2^2}{U}\left(\frac{\partial^2 u}{\partial z^2}\right)_{z=0} = -\frac{\delta_2^2 g}{\nu U}\frac{dh}{dr}, \tag{44}$$

where δ_2 is the momentum thickness and U the velocity at the edge of the boundary layer. Values of λ may be computed for the solution of §§ 3 and 4, and should give at any rate the order of magnitude of the true values, which would be derived from the solution of equation (43).

As noticed in § 4, h has a minimum value where $r = 2^{-\frac{1}{3}}l > r_0$. Consequently the pressure gradient is favourable for $r < 2^{-\frac{1}{3}}l$, and it suffices to consider the similarity solution of § 3. Here

$$\delta_2 = h\int_0^1 (f-f^2)\,d\eta = \frac{2(\pi - c\sqrt{3})}{3\sqrt{3}c^2}h,$$

so that

$$\lambda(r) = -\frac{4(\pi - c\sqrt{3})^2}{27c^4}\frac{gh^2}{\nu U}\frac{dh}{dr}$$

$$= -\frac{512\pi^{10}(\pi - c\sqrt{3})^2}{59049\sqrt{3}c^6}\frac{g\nu^3}{Q^5}\frac{(r^3+l^3)^3(r^3-\frac{1}{2}l^3)}{r^4}. \tag{45}$$

Since $d\lambda/dr < 0$, the worst value in any flow is that at the jump $r = r_1$.

The value of λ indicates the progress of the boundary layer towards separation, in many approximate solutions. A typical value at separation is $\lambda = -0{\cdot}082$ (Thwaites 1949), so it would be reasonable to neglect the gravitational pressure gradient if the values of $(-\lambda)$ derived from (45) were very small compared with 0·082.

7. Turbulent flow

In the analysis so far the motion in the layer has been assumed laminar. This may not always be realized, especially if the flow is unstable. The approximate formula of Lin (1945) for the velocity profile

$$u = U(r)f(\eta), \quad \eta = z/\delta,$$

gives the critical value for stability of

$$R_1 \equiv U\delta_1/\nu = 25f'(0)\int_0^1 (1-f(\eta))\,d\eta/f^4(\eta_c), \tag{46}$$

where η_c is determined from

$$-\pi f'(0)\,[f(\eta_c)f''(\eta_c)]/[f'(\eta_c)]^3 = 0{\cdot}58. \tag{47}$$

When $f(\eta)$ is the function of § 3, (47) becomes a cubic equation for $f'(\eta_c)$, namely

$$f'^3 + (3\pi c/1{\cdot}16)f'^2 - (3\pi c/1{\cdot}16) = 0.$$

Hence $f'(\eta_c) = 1{\cdot}327$, $f(\eta_c) = 0{\cdot}471$, and (46) leads to the critical value

$$R_1 = 275. \tag{48}$$

Now
$$R_1 = \left(1 - \frac{2\pi}{3\sqrt{3c^2}}\right)\frac{U\delta}{\nu}, \tag{49}$$

and thus from (34), when $r < r_0$
$$R_1 = \frac{3\sqrt{3c^2} - 2\pi}{3\sqrt{\{\pi c\sqrt{3}(\pi - c\sqrt{3})\}}}\left(\frac{Qr}{\nu a^2}\right)^{\frac{1}{2}}. \tag{50}$$

When $r \geqslant r_0$, $\delta = h$, and (22) gives
$$R_1 = \frac{3\sqrt{3c^2} - 2\pi}{4\pi^2}\frac{Q}{r\nu}. \tag{51}$$

Consequently the greatest value of R_1 occurs at $r = r_0$. For stability R_1 must be less than 275 everywhere, so the required condition is
$$R = \frac{Q}{a\nu} < \frac{8250\pi}{3\sqrt{3c^2} - 2\pi}\left(\frac{11\pi c\sqrt{3}(\pi - c\sqrt{3})}{3\sqrt{3c^2} - 2\pi}\right)^{\frac{1}{2}} = 2{\cdot}57 \times 10^4. \tag{52}$$

However, the stability criterion may be substantially modified by the fact that there is a free surface, so that (52) may be of little significance in practice.

If it is assumed that the motion is turbulent throughout, a solution for the flow within the jump may be obtained by introducing an eddy viscosity ϵ. In his treatment of the wall jet problem Glauert (1956) assumed that ϵ varies across the boundary layer like u^6, so as to agree near the wall with the Blasius law
$$\tau_w = 0{\cdot}0225\rho u^2(\nu/uz)^{\frac{1}{4}}, \tag{53}$$

where τ_w denotes the skin friction. As Glauert showed, if a similarity solution
$$u = U(r)\,F(\eta), \quad \eta = z/\delta(r), \tag{54}$$

exists, (53) requires that ϵ shall be of the form
$$\epsilon = \gamma U^{\frac{3}{4}}\delta^{\frac{3}{4}}F^6(\eta), \tag{55}$$

where γ is a constant. In the case of the wall jet Glauert used this variation of ϵ only in the region near the wall, where $\partial u/\partial z \geqslant 0$, since it was inappropriate to let $\epsilon \to 0$ as $z \to \infty$. Since $\partial u/\partial z \geqslant 0$ everywhere in the present problem, it seems reasonable to use the relation (55) throughout the layer, so that ϵ is greatest at the free surface.

In the similarity solution analogous to that of § 3, $\delta = h$ and
$$Q = 2\pi r\int_0^h u\,dz = 2\pi r U h\int_0^1 F(\eta)\,d\eta, \tag{56}$$

so that rUh is constant. The equation of motion
$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\epsilon\frac{\partial u}{\partial z}\right) \tag{57}$$

now becomes
$$\gamma(Uh)^{\frac{3}{4}}[d(F^6F')/d\eta] = h^2U'F^2.$$

Hence
$$U' = -\tfrac{9}{2}k^2\gamma h^{-\frac{5}{4}}U^{\frac{3}{4}}, \tag{58}$$

where k is a constant, so that
$$2[d(F^6F')/d\eta] = -9k^2F^2. \tag{59}$$

After multiplication by F^6F', (59) may be integrated to give

$$(F^6F')^2 = k^2(1-F^9), \tag{60}$$

since $F' = 0$ when $F = 1$. Thus

$$k\eta = \int_0^F \frac{x^6}{(1-x^9)^{\frac{1}{2}}}\,dx \tag{61}$$

and the condition $F(1) = 1$ gives

$$k = \int_0^1 \frac{x^6}{(1-x^9)^{\frac{1}{2}}}\,dx = \frac{1}{9}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{7}{9})}{\Gamma(\frac{23}{18})} = 0{\cdot}260. \tag{62}$$

Also
$$k\int_0^1 F(\eta)\,d\eta = \int_0^1 \frac{F^7}{(1-F^9)^{\frac{1}{2}}}\,dF = A, \tag{63}$$

where
$$A = \frac{1}{9}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{8}{9})}{\Gamma(\frac{25}{18})} = 0{\cdot}239. \tag{64}$$

Equation (56) therefore becomes

$$rUh = (k/2\pi A)\,Q. \tag{65}$$

The constant γ must be determined by the aid of (53). Since $F^6F' \to k$ as $\eta \to 0$, $k\eta \sim \frac{1}{7}F^7$. Also

$$\frac{\tau_w}{\rho} = \lim_{z\to 0}\left(\epsilon\frac{\partial u}{\partial z}\right) = \lim_{\eta\to 0}[\gamma(Uh)^{\frac{3}{4}}F^6UF'h^{-1}] = k\gamma U^{\frac{7}{4}}h^{-\frac{1}{4}}.$$

Equation (53) then leads to

$$k\gamma U^{\frac{7}{4}}h^{-\frac{1}{4}} = \lim_{\eta\to 0}[0{\cdot}0225U^{\frac{7}{4}}F^{\frac{7}{4}}\nu^{\frac{1}{4}}h^{-\frac{1}{4}}\eta^{-\frac{1}{4}}]$$

whence
$$k\gamma = 0{\cdot}0225(7k\nu)^{\frac{1}{4}}. \tag{66}$$

With h and γ eliminated by (65) and (66), (58) becomes

$$-\frac{U'}{U^2} = \frac{9}{2}(0{\cdot}0225)\,(7\nu)^{\frac{1}{4}}\left(\frac{2\pi Ar}{Q}\right)^{\frac{5}{4}}.$$

Consequently
$$U = \frac{100}{9(14)^{\frac{1}{4}}(\pi A)^{\frac{5}{4}}}\frac{Q^{\frac{5}{4}}}{\nu^{\frac{1}{4}}(r^{\frac{9}{4}}+l^{\frac{9}{4}})}, \tag{67}$$

and from (65)
$$h = \frac{9k}{200}(14\pi A)^{\frac{1}{4}}\left(\frac{\nu}{Q}\right)^{\frac{1}{4}}\frac{r^{\frac{9}{4}}+l^{\frac{9}{4}}}{r}, \tag{68}$$

where l is a constant length, which has to be determined by the conditions where the boundary layer reaches the free surface.

If the velocity profile (54) is assumed for $r < r_0$, with $U(r) = U_0$, the momentum integral equation, the analogue of (32), becomes

$$\left(\frac{d}{dr}+\frac{1}{r}\right)U_0^2\delta\left(\frac{A}{k}-\frac{2}{9k}\right) = 0{\cdot}0225(7k\nu)^{\frac{1}{4}}U_0^{\frac{7}{4}}\delta^{-\frac{1}{4}}.$$

This gives, on integration,

$$(r\delta)^{\frac{5}{4}} = \frac{k^{\frac{5}{4}}}{80(A-\frac{2}{9})}\left(\frac{7\nu}{U_0}\right)^{\frac{1}{4}} r^{\frac{9}{4}} + C,$$

where C is a constant. If the flow is turbulent throughout, and this equation applies down to $r = 0$, then $C = 0$. Thus

$$\delta = [80(A-\tfrac{2}{9})]^{-\frac{4}{5}}\, k(7\nu/U_0)^{\frac{1}{5}} r^{\frac{4}{5}}. \tag{69}$$

The volume flux through the boundary layer is $2\pi r U_0 \delta A/k$ and this becomes equal to Q at $r = r_0$. Hence

$$2\pi A U_0[80(A-\tfrac{2}{9})]^{-\frac{4}{5}}(7\nu/U_0)^{\frac{1}{5}} r_0^{\frac{9}{5}} = Q. \tag{70}$$

Then from (67), since $U = U_0$ when $r = r_0$,

$$l^{\frac{9}{4}} = 20(1-2A)\,(14\pi)^{-\frac{1}{4}} A^{-\frac{5}{4}}(Q/\nu)^{\frac{1}{4}} a^2. \tag{71}$$

When $r < r_0$ the total depth of the layer is

$$h = \frac{a^2}{2r} + \frac{k-A}{[80(A-\frac{2}{9})]^{\frac{4}{5}}}\left(\frac{7\nu}{U_0}\right)^{\frac{1}{5}} r^{\frac{4}{5}}. \tag{72}$$

Figure 3 shows the variation of U/U_0 and $(h/a)R^{\frac{1}{9}}$ as functions of $(r/a)R^{-\frac{1}{9}}$.

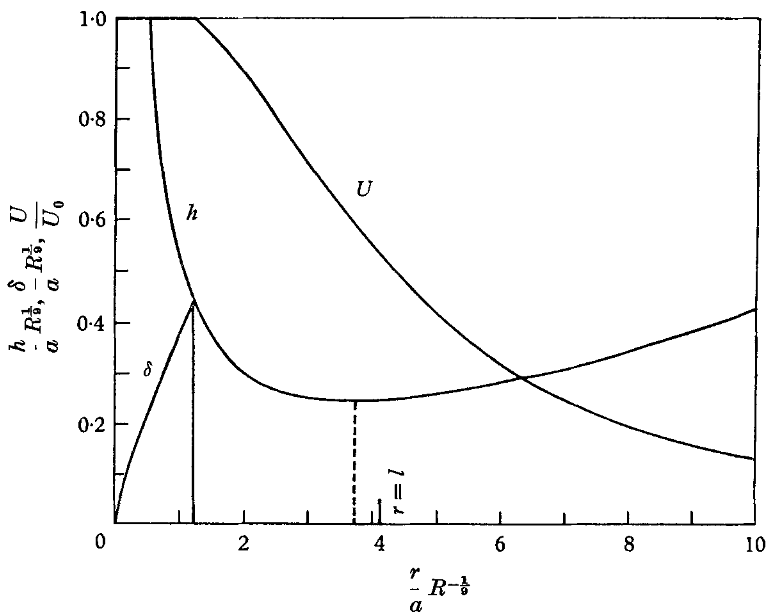

FIGURE 3. Variation of layer thickness and surface speed with radial distance (turbulent flow).

The jump condition is derived as in § 5. It is

$$\left.\begin{aligned} \frac{r_1 d^2 g a^2}{Q^2} + \frac{a^2}{2\pi^2 r_1 d} &= \frac{1}{\pi^2} - \frac{[560\pi(A-\frac{2}{9})]^{\frac{1}{5}}}{40\pi^2}\left(\frac{r_1}{a}R^{-\frac{1}{9}}\right)^{\frac{9}{5}} && (r_1 < r_0), \\ &= \frac{200}{81\,.\,14^{\frac{1}{4}}(\pi A)^{\frac{9}{4}}}\left[\left(\frac{r_1}{a}R^{-\frac{1}{9}}\right)^{\frac{9}{4}} + \frac{40(1-2A)}{(7\pi)^{\frac{1}{4}}(2A)^{\frac{5}{4}}}\right]^{-1} && (r_1 > r_0). \end{aligned}\right\} \tag{73}$$

8. Experiments

The experiments to be described were carried out in the Hydraulics department at the Engineering Laboratory, Cambridge. The plate used was a circular disk of plate glass, $\frac{3}{8}$ in. thick and 2 ft. in diameter, placed in a metal tray of the same diameter and supported on three screws arranged symmetrically 9 in. from the centre. Another screw at the centre of the tray was raised until it just touched the plate, in order to take the thrust of the jet. The tray had a rim of height 1 in. which held the glass plate from sliding horizontally, and the height of this rim above the plate could be adjusted by means of the screws. The tray stood on three legs, also adjustable by screws, on the floor of a large tank, and the tray and plate were levelled by means of the screws.

The jet was formed by a nozzle consisting of a circular hole drilled through a plug, which was screwed into a 2 in. pipe leading from the water main. The rate of flow could be varied by means of a valve in the 2 in. pipe. In order to reduce the effects of turbulence in the pipe, the plug was hollowed out to provide a smooth contraction from the pipe to the nozzle. The jet was arranged to strike the plate vertically at its centre.

In the experiments two sizes of nozzle were used, with diameters $\frac{1}{8}$ in. and $\frac{1}{4}$ in. The depth d outside the jump was governed principally by the height of the rim above the plate, and d was measured with a point gauge. The diameter of the standing wave was measured with dividers, and the rate of flow Q was found by measuring the quantity leaving the tank in a known period of time. The temperature of the water was also noted in order to find the viscosity. At small rates of flow the jet diameter was less than the nozzle diameter, and was therefore also measured.

The main difficulty found when taking measurements was the unsteadiness of the flow. The depth outside the jump was measured where it appeared to be greatest, slightly beyond the actual wave front, and had to be estimated by adjusting the point gauge until it was in and out of the water for equal amounts of time. The jump seldom formed a good circle about the jet, and a mean diameter was found and used for the calculations. At large rates of flow there was violent splashing from the point of impact of the jet, so that no observations could be made.

The measured values of r_1 ranged from under 1 in. to nearly 7 in., those of d from 0·13 in. to 0·65 in., and those of Q from 0·00043 to 0·0158 ft.^{3}/sec. The resultant jet Reynolds number $R = Q/a\nu$ varied from 7×10^3 to $1{\cdot}2 \times 10^5$. Figure 4 shows $\log_{10}[(r_1 d^2 g a^2/Q^2) + (a^2/2\pi^2 r_1 d)]$ plotted against $\log_{10}[(r_1/a) R^{-\frac{1}{3}}]$ for these observations, together with the curve obtained from the approximate solution and given by (41) and (42). Most of the observations appear to be reasonably consistent with this curve.

Following the suggestion of a referee that the discrepancies might be due to the neglect of the width of the jump, I have analysed the observations with respect to the ratio d/r_1. This ratio ranged between 0·02 and 0·5, and the analysis showed clearly that the points with $d/r_1 < 0{\cdot}1$ all lie near or above the curve, whereas those lying well below the curve correspond to large d/r_1, the worst

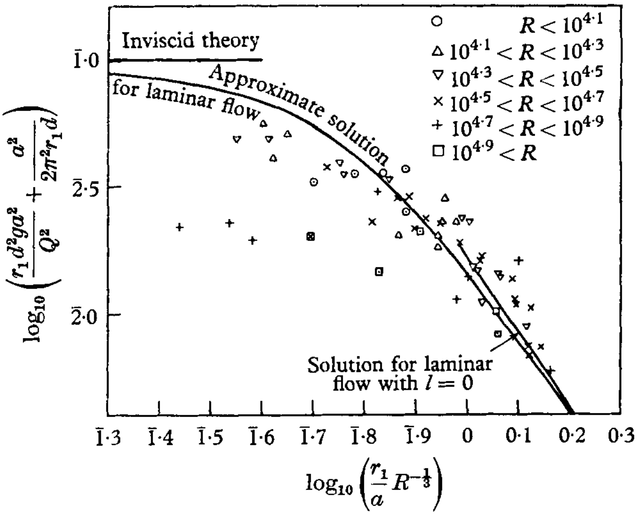

FIGURE 4. Comparison of experiment and theory for jump relation (laminar flow).

point having the greatest d/r_1. When allowance is made in the momentum equation for the width of the jump there is a skin friction force on the bottom of the fluid, and a greater area on which the pressure outside the jump can act. These effects both tend to make the experimental points fall below the theoretical curve, so that the results noted above can be explained qualitatively.

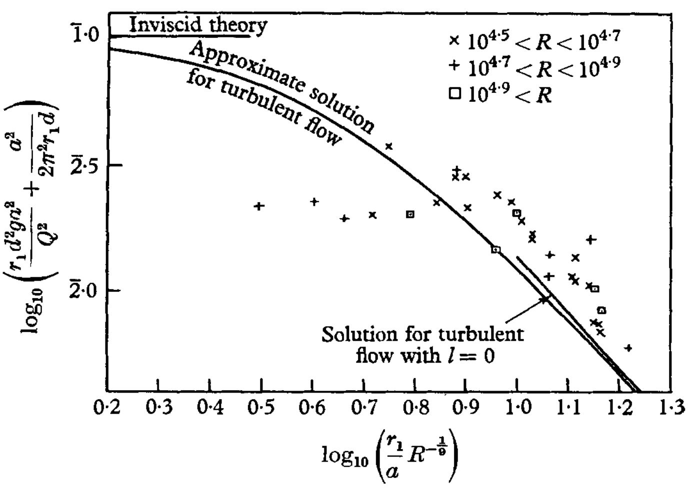

FIGURE 5. Comparison of experiment and theory for jump relation (turbulent flow).

Since in some observations the jet Reynolds number R was considerably greater than the critical value $2{\cdot}57 \times 10^4$ obtained by Lin's method, a comparison was also made with the theoretical result (73) for turbulent flow. This is given in figure 5, and is restricted to the observations with $R > 10^{4\cdot5}$. The agreement is less satisfactory than in figure 4.

From the measured quantities it was found that the greatest value of $-\lambda(r_1)$, obtained from (45), was 0·0015. Since this is less than 2% of the separation value 0·082, the assumption of negligible gravitational pressure gradient is well justified in these experiments.

It may be of some interest to compute the theoretical layer thickness and velocity for an experimental flow. The greatest jump radius attained was $r_1 = 6{\cdot}95$ in., and this was when $Q = 0{\cdot}00713$ ft.^{3}/sec, $\nu = 1{\cdot}23 \times 10^{-5}$ ft.^{2}/sec, $a = \frac{1}{8}$ in., and $d = 0{\cdot}135$ in. These values give $R = 5{\cdot}55 \times 10^4$, $(r_1/a)R^{-\frac{1}{3}} = 1{\cdot}46$, $[(r_1 d^2 g a^2/Q^2) + (a^2/2\pi^2 r_1 d)] = 0{\cdot}0059$, and $\lambda(r_1) = -0{\cdot}0015$. If the flow is assumed laminar, (24) gives $h = 0{\cdot}028$ in., and (23) gives $U = 1{\cdot}4$ ft./sec, both at the jump. Hence at the jump the depth increased by a factor of about 5. The least thickness of the layer was about 0·0076 in. where $r = 2{\cdot}15$ in. It might be difficult to measure such thicknesses experimentally, especially as the free surface often showed radial corrugations.

9. Two-dimensional flow

The corresponding problems in two-dimensional flow may readily be formulated, and the solutions are analogous. The flow might be realized by a two-dimensional jet striking a horizontal plane or, as suggested by Glauert (1956) for the wall jet, by the flow of water under a sluice gate. It will appear that the velocity profile in the similarity solution is the same as in the axisymmetric case. This applies both for laminar and for turbulent flow, and the result also holds for the wall jet and the radial free jet.

If x, y are rectangular co-ordinates with y vertically upwards, and u, v are the corresponding velocity components, the analogues of equations (8) to (12) for laminar flow are

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{74}$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = \nu(\partial^2 u/\partial y^2), \tag{75}$$

$$u = v = 0 \quad \text{at} \quad y = 0, \tag{76}$$

$$\partial u/\partial y = 0 \quad \text{at} \quad y = h(x), \tag{77}$$

$$\int_0^{h(x)} u\,dy = Q. \tag{78}$$

Here Q is the flow per unit span in the positive x-direction. Thus the total flow from the two-dimensional jet would be $2Q$, and the flow under the sluice gate Q.

The similarity solution of these equations, derived as in § 3, is

$$u = U(x) f(\eta), \quad \eta = y/h(x), \tag{79}$$

where $f(\eta)$ is the function defined by equations (19) or (25), and

$$U(x) = (9c^2/2\pi^2)\, Q^2/\nu(x+l), \tag{80}$$

$$h(x) = (\pi/\sqrt{3})\, \nu(x+l)/Q. \tag{81}$$

Here l is an arbitrary constant length, whose presence merely indicates that a shift of origin is possible.

In order to develop an approximate general solution corresponding to that of § 4, the speed at the edge of the boundary layer is taken as the constant U_0 for $0 < x < x_0$. If the flow were realized physically by one of the examples mentioned above U_0 would be the speed of the impinging jet, or the speed attained by the flow under the sluice gate a short distance downstream. The corresponding inviscid flow has the uniform depth $a = Q/U_0$. The characteristic Reynolds number is now

$$R = U_0 a/\nu = Q/\nu. \tag{82}$$

If the velocity profile is assumed constant, of the form

$$u = U_0 f(\eta), \quad \eta = y/\delta(x), \tag{83}$$

where $f(\eta)$ is again the function of § 3, substitution in the momentum integral equation leads to the result

$$\delta^2(x) = \frac{3\sqrt{3}c^3}{2(\pi - c\sqrt{3})}\frac{\nu x}{U_0}, \tag{84}$$

assuming $\delta = 0$ at $x = 0$. The total thickness of the layer is

$$h = a + (1 - 2\pi/3\sqrt{3}c^2)\,\delta. \tag{85}$$

Since $\delta = h$ when $x = x_0$,

$$x_0 = \{3\sqrt{3}c(\pi - c\sqrt{3})/2\pi^2\}\,aR. \tag{86}$$

The value of the length l in (80) and (81) can now be estimated as

$$l = \{3\sqrt{3}c(2\sqrt{3}c - \pi)/2\pi^2\}\,aR. \tag{87}$$

The use of the principles of momentum and continuity at the jump now gives the position $x = x_1$ of the jump. When $(h/d)^2$ is neglected as before,

$$\left.\begin{aligned} \frac{x_1}{aR} &= \frac{3\sqrt{3}c}{2(\pi - c\sqrt{3})}\left(1 - \frac{gd^2a}{2Q^2} - \frac{a}{d}\right)^2 && \text{for} \quad x_1 < x_0, \\ &= \frac{9\sqrt{3}c^3}{2\pi^3}\left(\frac{gd^2a}{2Q^2} + \frac{a}{d}\right)^{-1} - \frac{3\sqrt{3}c(2\sqrt{3}c - \pi)}{2\pi^2} && \text{for} \quad x_1 > x_0. \end{aligned}\right\} \tag{88}$$

The inviscid theory, analogous to that of § 2, is identical with Rayleigh's (1914) theory, when the depth h is regarded as constant and equal to a. It leaves the position of the jump indeterminate but gives

$$\frac{gd^2a}{2Q^2}\left(1 + \frac{a}{d}\right) = 1, \tag{89}$$

or, if the pressure thrust ahead of the wave is neglected as in (88),

$$\frac{gd^2a}{2Q^2} + \frac{a}{d} = 1. \tag{90}$$

Thus (88) shows that if the left-hand side of (90) is less than 1, the flow loses total head by friction over the length x_1 until the jump can occur.

The analysis for two-dimensional turbulent flow is analogous to that of § 7. The similarity solution is

$$u = U(x)\,F(\eta), \quad \eta = y/h(x), \tag{91}$$

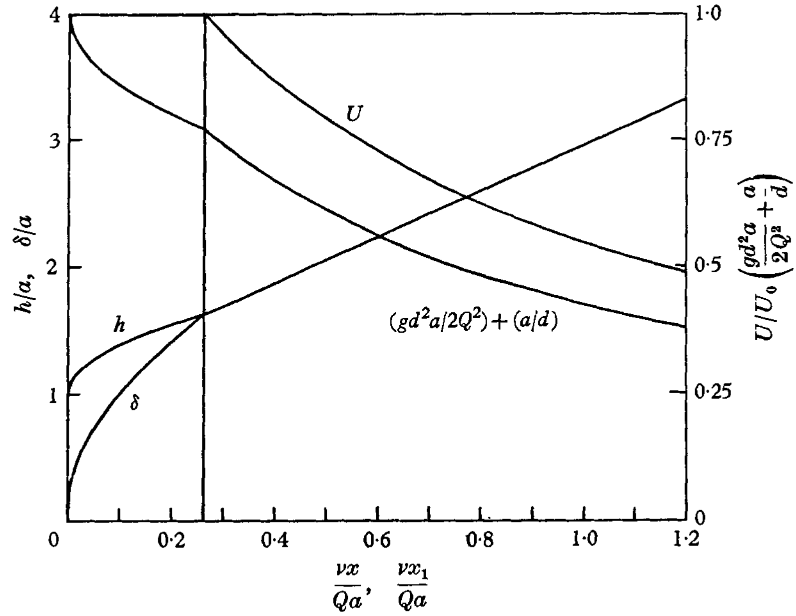

FIGURE 6. Variation of layer thickness and surface speed with distance, and jump relation, in two-dimensional laminar flow.

where $F(\eta)$ is the function defined by (61), and

$$U(x) = \frac{800}{81 \times 7^{\frac{1}{4}} A^{\frac{5}{4}}} \frac{Q^{\frac{5}{4}}}{\nu^{\frac{1}{4}}(x+l)}, \tag{92}$$

$$h(x) = \frac{81(7A)^{\frac{1}{4}} k}{800} \left(\frac{\nu}{Q}\right)^{\frac{1}{4}} (x+l). \tag{93}$$

The corresponding approximate general solution gives, for $x < x_0$,

$$\delta(x) = \left(\frac{81}{320\,(9A-2)}\right)^{\frac{4}{5}} 7^{\frac{1}{5}} k \left(\frac{a\nu}{Q}\right)^{\frac{1}{5}} x^{\frac{4}{5}}, \tag{94}$$

where δ is the boundary-layer thickness, and the total thickness is

$$h = a + \left(1 - \frac{A}{k}\right) \delta. \tag{95}$$

Also

$$x_0 = \frac{320(9A-2)}{81 \times 7^{\frac{1}{4}} A^{\frac{5}{4}}} aR^{\frac{1}{4}}, \tag{96}$$

and the value of l in (92) and (93) is found as

$$l = \frac{160(1-2A)}{9 \times 7^{\frac{1}{4}} A^{\frac{5}{4}}} aR^{\frac{1}{4}}. \tag{97}$$

The jump condition now becomes

$$\left.\begin{aligned} \frac{x_1}{a} R^{-\frac{1}{4}} &= \frac{320}{9} (7A - \tfrac{14}{9})^{-\frac{1}{4}} \left(1 - \frac{gd^2a}{2Q^2} - \frac{a}{d}\right)^{\frac{5}{4}} && \text{for} \quad x_1 < x_0, \\ &= \frac{1600}{729 \times 7^{\frac{1}{4}} A^{\frac{9}{4}}} \left(\frac{gd^2a}{2Q^2} + \frac{a}{d}\right)^{-1} - \frac{160(1-2A)}{9 \times 7^{\frac{1}{4}} A^{\frac{5}{4}}} && \text{for} \quad x_1 > x_0. \end{aligned}\right\} \tag{98}$$

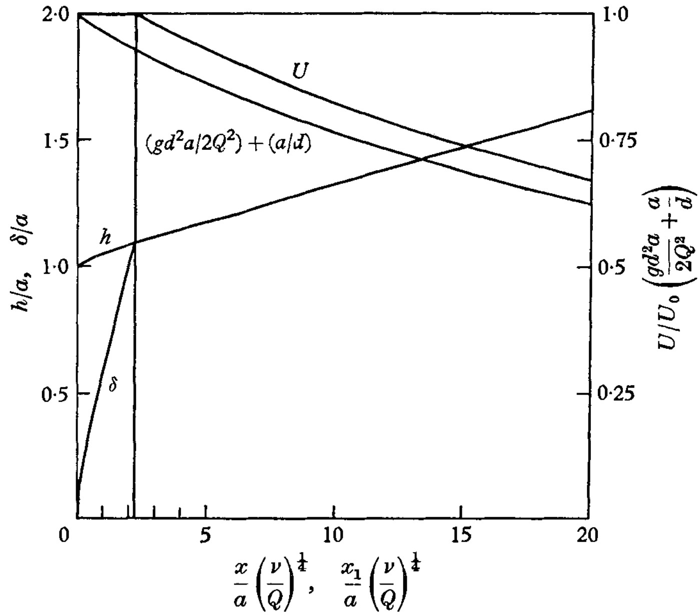

FIGURE 7. Variation of layer thickness and surface speed with distance, and jump relation, in two-dimensional turbulent flow.

The experimental work was carried out in the Hydraulics department at the Engineering Laboratory, Cambridge, by the kindness of Prof. J. F. Baker. I am greatly indebted to Mr A. M. Binnie, whose advice and encouragement enabled me to perform the experiments. To the many persons who have kindly interested themselves in this work I offer my apologies for the inordinate delay in its presentation.

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